A Toy Model for the Magnetic Connection
between a Black Hole and a Disk

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A magnetic field connecting a Kerr black hole to a disk rotating around it can extract energy and angular momentum from the black hole and transfer them to the disk if the black hole rotates faster than the disk. The energy can be dissipated and radiated away by the disk, which makes the disk shine without the need of accretion. In this paper we present a toy model for the magnetic connection: a single electric current flowing around a Kerr black hole in the equatorial plane generates a poloidal magnetic field which connects the black hole to the disk. The rotation of the black hole relative to the disk generates an electromotive force which in turn generates a poloidal electric current flowing through the black hole and the disk and produces a power on the disk. We will consider two cases: (1) The toroidal current flows on the inner boundary of the disk, which generates a poloidal magnetic field connecting the horizon of the black hole to a region of the disk *beyond* the inner boundary; (2) The toroidal current flows on a circle inside the inner boundary of the disk but outside the horizon of the black hole, which generates a poloidal magnetic field connecting a portion of the horizon of the black hole to the *whole* disk. We will calculate the power produced by the magnetic connection and the resulting radiation flux of the disk in the absence of accretion, and compare them with that produced by accretion.

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I.   INTRODUCTION

A magnetic field connecting a Kerr black hole to a disk rotating around it can transfer energy and angular momentum between the black hole and the disk ([1, 2, 3, 4] and references therein). When the black hole rotates faster than the disk, energy and angular momentum are extracted from the black hole and transfered to the disk. If the magnetic field is weak, the energy transfered to the disk by the magnetic connection can be dissipated and radiated away by the disk [2, 3, 4]. Thus, in the presence of magnetic connection, the rotational energy of the black hole provides an energy source for disk radiation in addition to disk accretion [5, 6, 7]. In fact, a disk magnetically coupled to a rapidly rotating Kerr black hole can radiate without the need of accretion [2, 3, 4]. In such a case, all the power of the disk comes from the rotational energy of the black hole.

Due to the lack of knowledge about the topology and distribution of the magnetic field connecting the black hole to the disk, a detailed model of the magnetic connection between a black hole and a disk does not exist yet, except the simplest and *ad hoc* model where the magnetic field is assumed to connect the black hole to the disk close to the inner boundary [2, 3], or at a single radius beyond the inner boundary [4]. In this paper, we consider a toy model for the magnetic connection: A single toroidal electric current flows around a Kerr black hole in

the equatorial plane which is coincident with the plane of a thin Keplerian disk rotating around the black hole. The toroidal current generates a poloidal magnetic field which connects the black hole to the disk. Due to the relative rotation between the black hole and the disk and the fact that the horizon of a Kerr black hole is a conductor with a finite electric resistivity [8, 9, 10] and a plasma disk is a perfect conductor, an electromotive forces (EMF) is induced which in turn generates a poloidal electric current flowing through the black hole and the disk. The poloidal current produces a torque and a power on the disk, which makes the disk shine without the need of accretion. We will calculate the power and the resulting radiation flux of the disk with the assumption that there is no accretion and the configuration is steady and axisymmetric, and compare the results with that of a standard accretion disk.

A similar model has been used by Li [11] to test the efficiency of the Blandford-Znajek mechanism [12, 13, 14]. By assuming a toroidal current continuously distributed in a thin Keplerian disk, Li [11] has shown that the power of the disk always dominates over the power of the black hole, which confirms the early speculation of Blandford and Znajek [12] and the recent arguments of Livio, Ogilvie, and Pringle [15]. In this paper we will find that, an electric current flowing around a black hole in the equatorial plane naturally generates a magnetic field connecting the black hole to its disk. This implies that the existence of magnetic connection between a black hole and a disk is a very natural assumption in view of physics. Essentially the magnetic connection between a black hole and a disk is a variant of the Blandford-Znajek mechanism. However, it is simpler and better defined than the Blandford-Znajek mechanism since a unknown

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remote load is not required [4]. Though the model presented in this paper is so simple that it cannot be directly applied to real astronomy, it helps us in understanding some physical aspects of the magnetic connection.

Throughout the paper we use the geometrized units $G = c = 1$ and the Boyer-Lindquist coordinates (t, r, θ, ϕ) [16, 17].

II. THE MAGNETIC FLUX AND THE MAGNETIC CONNECTION

Assuming a neutral toroidal electric current I flows on a circle of $r = r'$ in the equatorial plane of a Kerr black hole, then the four-current vector is

$$J^a = \frac{I}{r} \left(\frac{\Delta}{A} \right)^{1/2} \left(\frac{\partial}{\partial \phi} \right)^a \delta(r - r') \delta(\cos \theta), \quad (1)$$

where $\Delta \equiv r^2 - 2Mr + a^2$, $A \equiv (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta$, M is the mass of the black hole, Ma is the angular momentum of the black hole $(a^2 \leq M^2)$, and $\delta(x)$ is the Dirac δ-function. The magnetic flux through a surface bounded by a circle with $r = $ constant and $\theta = $ constant, produced by such an electric current, is

$$\Psi(r, \theta; r') = 2\pi A_\phi(r, \theta; r'), \quad (2)$$

where A_ϕ, the toroidal component of the electric vector potential, is given by [18, 19]

$$
\begin{aligned}
A_\phi &= 2 \sum_{l=1}^{\infty} \left\{ \alpha_l^r \left[ra \sin^2 \theta \frac{\Delta}{\Sigma} \frac{1}{\sqrt{M^2 - a^2}} P_l'(u) P_l(\cos \theta) - a \sin^2 \theta \cos \theta \frac{r^2 + a^2}{\Sigma} P_l(u) P_l'(\cos \theta) \right] \right. \\
&\quad + \alpha_l^i \left[-a^2 \sin^2 \theta \cos \theta \frac{\Delta}{\Sigma} \frac{1}{\sqrt{M^2 - a^2}} P_l'(u) P_l(\cos \theta) - r \sin^2 \theta \frac{r^2 + a^2}{\Sigma} P_l(u) P_l'(\cos \theta) \right. \\
&\quad \left. \left. + \frac{\Delta \sin^2 \theta}{l(l+1)} \frac{1}{\sqrt{M^2 - a^2}} P_l'(u) P_l'(\cos \theta) \right] \right\} \\
&\quad + 2 \sum_{l=1}^{\infty} \left\{ \beta_l^r \left[ra \sin^2 \theta \frac{\Delta}{\Sigma} \frac{1}{\sqrt{M^2 - a^2}} Q_l'(u) P_l(\cos \theta) - a \sin^2 \theta \cos \theta \frac{r^2 + a^2}{\Sigma} Q_l(u) P_l'(\cos \theta) \right] \right. \\
&\quad + \beta_l^i \left[-a^2 \sin^2 \theta \cos \theta \frac{\Delta}{\Sigma} \frac{1}{\sqrt{M^2 - a^2}} Q_l'(u) P_l(\cos \theta) - r \sin^2 \theta \frac{r^2 + a^2}{\Sigma} Q_l(u) P_l'(\cos \theta) \right. \\
&\quad \left. \left. + \frac{\Delta \sin^2 \theta}{l(l+1)} \frac{1}{\sqrt{M^2 - a^2}} Q_l'(u) P_l'(\cos \theta) \right] \right\},
\end{aligned}
\quad (3)
$$

where $u \equiv (r - M)/\sqrt{M^2 - a^2}$, $P_l(z)$ and $Q_l(z)$ are Legendre functions, $P_l'(z) \equiv dP_l(z)/dz$, $Q_l'(z) \equiv dQ_l(z)/dz$, and the coefficients α_l^r, α_l^i, β_l^r, and β_l^i are respectively (1) for $r < r'$, $\beta_l^r = \beta_l^i = 0$ for all l; but

$$\alpha_l^r \equiv \frac{\pi(2l+1)I}{l(l+1)(M^2 - a^2)} \left(\frac{\Delta'}{A'} \right)^{1/2} \Delta' a P_l(0) Q_l'(u'), \quad (4)$$

$$
\begin{aligned}
\alpha_l^i &\equiv \frac{\pi(2l+1)I}{l(l+1)\sqrt{M^2 - a^2}} \left(\frac{\Delta'}{A'} \right)^{1/2} \left[-(r'^2 + a^2) P_l'(0) \right. \\
&\quad \left. \times Q_l(u') + \frac{r'\Delta'}{l(l+1)} \frac{1}{\sqrt{M^2 - a^2}} P_l'(0) Q_l'(u') \right]; \quad (5)
\end{aligned}
$$

(2) for $r > r'$, $\alpha_l^r = \alpha_l^i = 0$ for all l; but

$$\beta_l^r \equiv \frac{\pi(2l+1)I}{l(l+1)(M^2 - a^2)} \left(\frac{\Delta'}{A'} \right)^{1/2} \Delta' a P_l(0) P_l'(u'), \quad (6)$$

$$
\begin{aligned}
\beta_l^i &\equiv \frac{\pi(2l+1)I}{l(l+1)\sqrt{M^2 - a^2}} \left(\frac{\Delta'}{A'} \right)^{1/2} \left[-(r'^2 + a^2) P_l'(0) \right. \\
&\quad \left. \times P_l(u') + \frac{r'\Delta'}{l(l+1)} \frac{1}{\sqrt{M^2 - a^2}} P_l'(0) P_l'(u') \right]; \quad (7)
\end{aligned}
$$

where $\Delta' = \Delta(r = r')$ and $A' = A(r = r', \theta = \pi/2)$.

The magnetic flux through the hemi-sphere of the horizon of the black hole is $\Psi(r_H, \pi/2; r')$, where $r_H = M + \sqrt{M^2 - a^2}$ is the radius of the horizon. The magnetic flux through a surface bounded by a circle $r = $ constant in the equatorial plane is $\Psi(r, \pi/2; r')$. The magnetic field lines generated by the toroidal current are closed loops around the circle $r = r'$, except the field line along the axis $\theta = 0$. Consider the field lines leaving the horizon at a polar angle θ_1, they will intersect the equatorial plane at a circle of radius r_1 where r_1 is given by $\Psi(r_1, \pi/2; r') = \Psi(r_H, \theta_1; r')$. Clearly, if the inner edge of the disk lying in the equatorial plane is inside the cir-
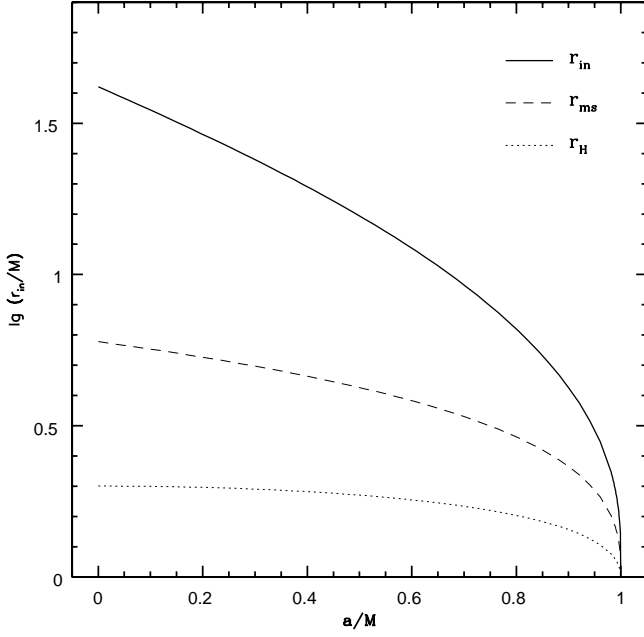
FIG. 1: The radius where the innermost flux surface intersects the disk as a function of the spin of the black hole (the solid line): the toroidal electric current flows on the marginally stable orbit $r' = r_{ms}$. For comparison, the inner edge of the disk (the dashed line) and the horizon of the black hole (the dotted line) are also shown.
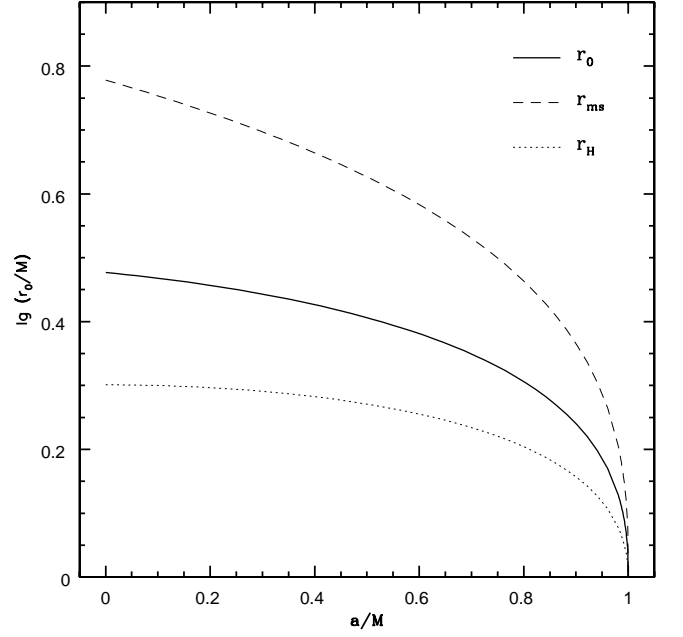

FIG. 2: The radius where if the toroidal current is put then the generated magnetic field connects the portions of the horizon of the black hole with $0 < \theta < \pi/3$ and $2\pi/3 < \theta < \pi$ to the whole disk from $r = r_{ms}$ to $r = \infty$ (the solid line). For comparison, the inner edge of the disk (the dashed line) and the horizon of the black hole (the dotted line) are also shown.

cle $r = r_1$ and the outer edge of the disk is outside the circle $r = r_1$, a magnetic connection must exist between the black hole and the disk.

Let us consider a thin Keplerian disk in the equatorial plane of a Kerr black hole [5, 6, 7]. The angular velocity of the disk is

$$\Omega_D(r) = \left(\frac{M}{r^3}\right)^{1/2} \frac{1}{1 + a\left(\frac{M}{r^3}\right)^{1/2}} . \quad (8)$$

The outer edge of the disk is at $r = \infty$. The inner edge of the disk is at the marginally stable orbit

$$r_{ms} = M\left\{3 + z_2 - [(3 - z_1)(3 + z_1 + 2z_2)]^{1/2}\right\} \quad (9)$$

where

$$z_1 \equiv 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/3}$$
$$\times \left[\left(1 + \frac{a}{M}\right)^{1/3} + \left(1 - \frac{a}{M}\right)^{1/3}\right] , \quad (10a)$$

$$z_2 \equiv \left(z_1^2 + 3\frac{a^2}{M^2}\right)^{1/2} . \quad (10b)$$

Due to the existence of a gap between the horizon of the black hole and the inner boundary of the the disk, not all magnetic field lines connect the black hole to the disk. If we define a flux surface to be a surface of constant magnetic flux, there is an *innermost flux surface*

which connects the black hole to the disk. Assuming the toroidal electric current is located at the inner boundary of the disk: $r' = r_{ms}$, then the innermost flux surface touches the horizon of the black hole at $\theta = \pi/2$ and intersects the disk at a circle whose radius r_{in} is determined by

$$\Psi\left(r_{in}, \frac{\pi}{2}; r_{ms}\right) = \Psi\left(r_H, \frac{\pi}{2}; r_{ms}\right) , \quad r_{in} > r_{ms} . \quad (11)$$

Since the magnetic field lines in the gap between the black hole and the disk must close themselves on the disk, r_{in} must be greater than r_{ms}. Clearly, r_{in}/M depends only on a/M when $r' = r_{ms}$ is fixed. In Fig. 1 we plot r_{in}/M as a function of a/M. For comparison, we also plot r_{ms}/M and r_H/M in Fig. 1. We see that, as a/M increases, r_{in} gets closer to r_{ms}. But, for $a/M < 1$, we always have $r_{in} > r_{ms}$. The magnetic field lines between r_{ms} and r_{in} close themselves in the gap region between r_H and r_{ms}, they do not touch the black hole horizon.

In order for the magnetic field to connect the black hole to the whole disk from $r = r_{ms}$ to $r = \infty$, the toroidal current must be inside the inner edge of the disk but outside the horizon of the black hole. Assume $r' = r_0$ when the inner most flux surface touches the disk at $r = r_{ms}$ and the horizon of the black hole at $\theta = \pi/3$ (and $\theta = 2\pi/3$ due to the reflection symmetry about the equatorial plane). Then, r_0 is determined by

$$\Psi\left(r_{ms}, \frac{\pi}{2}; r_0\right) = \Psi\left(r_H, \frac{\pi}{3}; r_0\right) , \quad r_H < r_0 < r_{ms} . \quad (12)$$
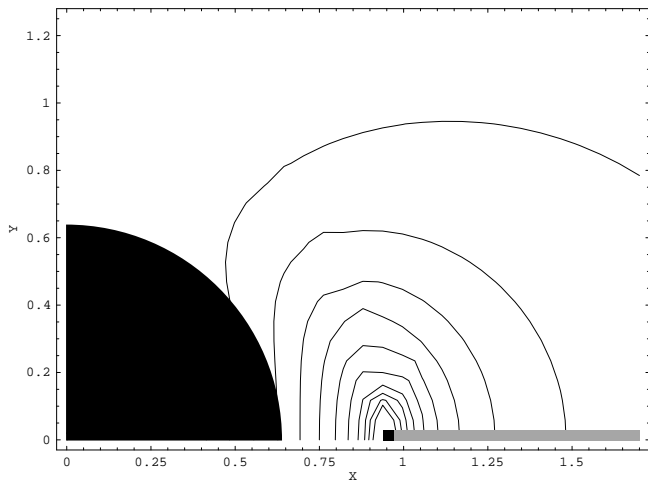
FIG. 3: The topology of the magnetic field produced by a toroidal electric current flowing on the marginally stable orbit in the equatorial plane of a Kerr black hole of mass M and specific angular momentum $a = 0.99M$. The horizontal axis is $X \equiv (r/M)\cos\theta - 0.5$, the vertical axis is $Y \equiv (r/M)\sin\theta - 0.5$. The black disk represents the inside of the black hole, whose boundary is the horizon of the black hole. The gray line lying on the X-axis is the Keplerian disk. The black dot represents the inner edge of the disk which is at the marginally stable orbit. The solid curves represent the magnetic field lines (tangent to the flux surfaces) generated by the toroidal current. The innermost flux surface touches the horizon at $\theta = 0$ and touches the disk at a radius beyond the inner boundary.

Like r_{in}/M, r_0/M also depends only on a/M. In Fig. 2 we plot r_0/M as a function of a/M. For comparison, we also plot r_{ms}/M and r_H/M in Fig. 2.

Examples of the topological structure of the magnetic field produced by a toroidal electric current flowing in the equatorial plane of a Kerr black hole are shown in Fig. 3 and Fig. 4.

III. THE POWER PRODUCED BY THE MAGNETIC CONNECTION

In the presence of a magnetic field a Kerr black hole behaves like a conductor with a surface resistivity $R_H = 4\pi \approx 377$ Ohms [8, 9, 10]. A plasma disk can be treated as a perfect conductor with a vanishing resistivity. So, the rotation of the black hole and the disk induces an EMF on the horizon of the black hole and an EMF on the disk [11, 13]. For a bunch of magnetic field lines connecting the black hole to the disk within radii r — $r + \Delta r$, where $\Delta r \ll r$, the induced EMFs on the horizon of the black hole and the disk are respectively

$$\Delta\mathcal{E}_H = \frac{1}{2\pi}\Omega_H\Delta\Psi, \quad \Delta\mathcal{E}_D = -\frac{1}{2\pi}\Omega_D\Delta\Psi, \quad (13)$$
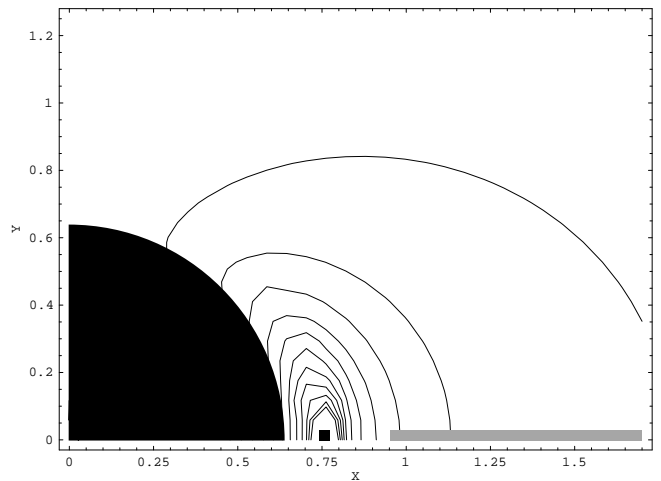

FIG. 4: The topology of the magnetic field produced by a toroidal electric current flowing on a circle in the equatorial plane. The model is the same as that in Fig. 1 except that now the electric current is inside the inner boundary of the disk. The innermost flux surface touches the horizon of the black hole at $\theta = \pi/3$ (and $\theta = 2\pi/3$) and touches the disk at the inner boundary.

where $\Delta\Psi$ is the magnetic flux connecting the black hole to the disk, Ω_H is the angular velocity of the black hole

$$\Omega_H = \frac{a}{2Mr_H}, \quad (14)$$

which is constant over the horizon of the black hole. For a thin Keplerian disk the angular velocity of the disk is given by Eq. (8).

The black hole and the disk form a closed electric circuit, an electric current flows along the magnetic field lines connecting the black hole to the disk and closes itself inside the disk and the black hole. We assume that there is a thin plasma corona around the black hole and the disk, which can carry the poloidal current flowing between the black hole and the disk [12]. In the magnetosphere the resistivity along the magnetic field lines is negligible, while the resistivity perpendicular to the magnetic field lines is large. Thus the electric current flows along the magnetic field lines in the corona without dissipation. Suppose the disk and the black hole rotates in the same direction, then the EMF of the black hole, $\Delta\mathcal{E}_H$, and the EMF of the disk, $\Delta\mathcal{E}_D$, have opposite signs. The direction of the electric current, and thus the direction of the transfer of energy and angular momentum, is determined by the sign of $\Delta\mathcal{E}_H + \Delta\mathcal{E}_D$. If $\Delta\mathcal{E}_H + \Delta\mathcal{E}_D > 0$, the EMF of the black hole dominates the EMF of the disk, so the black hole "charges" the disk: energy and angular momentum are transfered from the black hole to the disk. On the other hand, if $\Delta\mathcal{E}_H + \Delta\mathcal{E}_D < 0$, the EMF of the disk dominates the EMF of the black hole, so the disk "charges" the black hole, energy and angular momentum are transfered from the disk to the black hole [3].
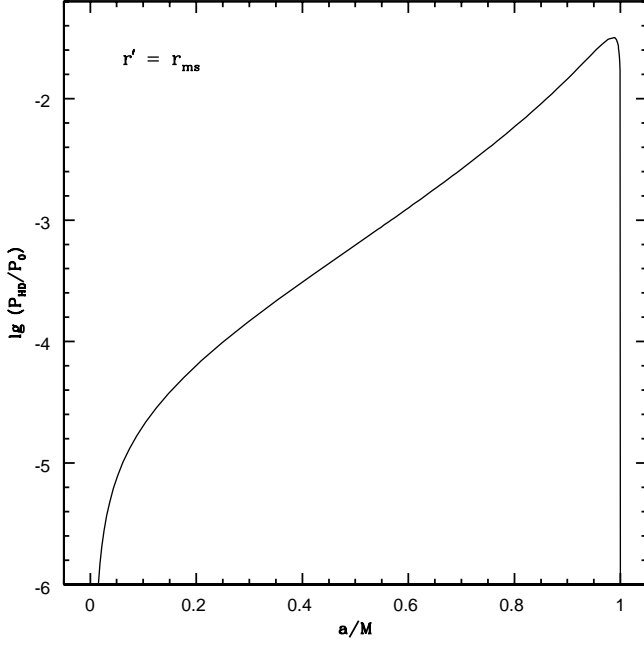
FIG. 5: The power produced by the magnetic connection as a function of the black hole spin: The magnetic field produced by a toroidal electric current flowing on the inner edge of the disk connects the horizon of the black hole to the regions of the disk with $r > r_{in}$, where r_{in} ($> r_{ms}$, where r_{ms} is the radius of the marginally stable orbit where the inner boundary of the disk is) is defined by Eq. (11). The power peaks at $a/M = 0.986$ and drops to zero as a/M approaches 1.
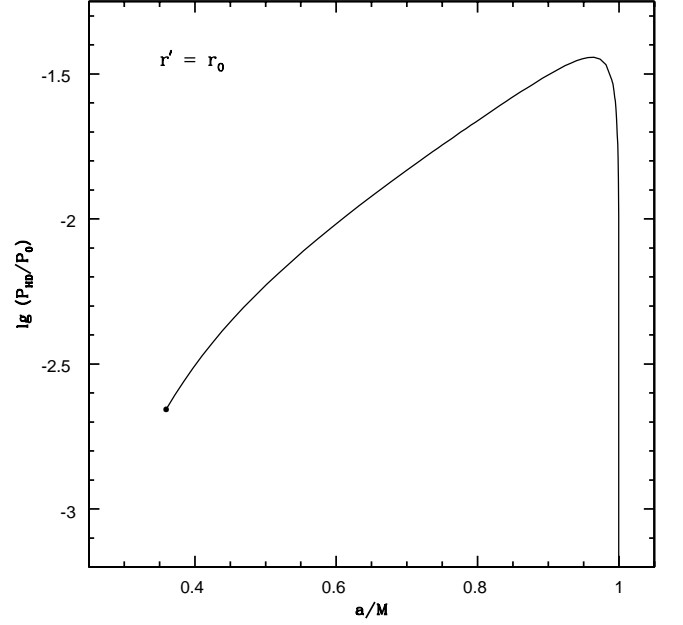

FIG. 6: The power produced by the magnetic connection as a function of the black hole spin: The magnetic field produced by a toroidal electric current flowing on a circle of radius r_0 [defined by Eq. (12)] in the equatorial plane connects the portions of the horizon of the black hole with $0 < \theta < \pi/3$ and $2\pi/3 < \theta < \pi$ to the whole disk from $r = r_{ms}$ to $r = \infty$. The power peaks at $a/M = 0.961$ and drops to zero as a/M approaches 1. The thick point is at $a/M = 0.3594$ below which steady state solutions do not exist.

An infinite number of adjacent infinitesimal poloidal electric current loops flow between the black hole and the disk along the magnetic field lines connecting them. Each infinitesimal current loop produces an infinitesimal torque and an infinitesimal power on the disk [2]

$$\Delta T_{HD} = \left(\frac{\Delta \Psi}{2\pi}\right)^2 \frac{\Omega_H - \Omega_D}{\Delta Z_H}, \qquad (15)$$

$$\Delta P_{HD} = \Delta T_{HD} \Omega_D, \qquad (16)$$

where ΔZ_H is the resistance of the black hole associated with the infinitesimal current. In Eq. (16) we have used the fact that the disk is perfectly conducting, so the magnetic field lines are frozen in and corotate with the disk. The summation of all ΔT_{HD} gives the the total torque on the disk, the summation of all ΔP_{HD} gives the the total power on the disk. In the limit $\Delta r \to 0$ and $\Delta \Psi \to 0$, the summation is replaced by integration. Then, after taking into account that a disk has two surfaces, we obtain the total torque and the total power on the disk

$$T_{HD} = 4\pi \int_{r_a}^{\infty} Hr dr, \quad P_{HD} = 4\pi \int_{r_a}^{\infty} \Omega Hr dr, \quad (17)$$

where r_a is the radius in the disk where the magnetic connection starts ($r_a = r_{in}$ for $r' = r_{ms}$; $r_a = r_{ms}$ for

$r' = r_0$),

$$H \equiv \frac{1}{8\pi^2 r} \left(\frac{d\Psi}{dr}\right)^2 \frac{\Omega_H - \Omega_D}{-dZ_H/dr}, \qquad (18)$$

where we have treated the resistance of the black hole, Z_H, as a function of the radius of the disk, which is defined by a map from the horizon of the black hole to the surface of the disk given by the magnetic field lines connecting them.

For a Kerr black hole we have [13]

$$\frac{dZ_H}{d\theta} = \frac{R_H}{2\pi} \frac{r_H^2 + a^2 \cos^2 \theta}{(r_H^2 + a^2) \sin \theta}, \qquad (19)$$

where θ is the polar angle coordinate on the horizon. Then, dZ_H/dr can be calculated through

$$\frac{dZ_H}{dr} = \frac{dZ_H}{d\theta} \frac{d\theta}{dr}, \qquad (20)$$

where $\theta = \theta(r)$ is a map between the θ coordinate on the horizon and the r coordinate on the disk, which is induced by the magnetic field lines connecting the disk to the horizon

$$\Psi\left(r_H, \theta; r'\right) = \Psi\left(r, \frac{\pi}{2}; r'\right). \qquad (21)$$
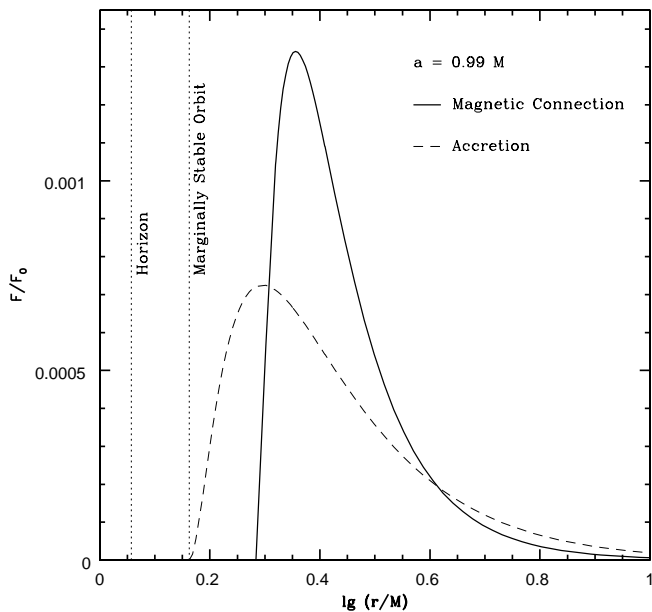
FIG. 7: The radiation flux of a non-accretion disk magnetically coupled to a Kerr black hole of mass M and specific angular momentum $a = 0.99M$: The poloidal magnetic field is generated by a toroidal electric current flowing on the inner boundary of the disk which is at the marginally stable orbit. The magnetic field connects the horizon of the black hole to the disk in the regions with $r > r_{in}$ where r_{in} is defined by Eq. (11). The dashed curve is the radiation flux of a standard accretion disk around the same black hole.

FIG. 8: The radiation flux of a non-accretion disk magnetically coupled to a Kerr black hole of mass M and specific angular momentum $a = 0.99M$: The poloidal magnetic field is generated by a toroidal electric current flowing on a circle of radius r_0 inside the inner boundary of the disk, where r_0 is defined by Eq. (12). The magnetic field connects the portions of the horizon of the black hole with $0 < \theta < \pi/3$ and $2\pi/3 < \theta < \pi$ to the disk from $r = r_{ms}$ to $r = \infty$. The dashed curve is the radiation flux of a standard accretion disk around the same black hole.

Since $d\theta/dr < 0$ and $dZ_H/d\theta > 0$, we have $dZ_H/dr < 0$.

For a given toroidal current loop at $r = r'$, we can calculate the magnetic flux with Eqs. $(2 - 7)$, determine the innermost flux surface and thus r_a, determine the map defined by Eq. (21). Then, with Eq. (17) and Eq. (18), we can calculate the power of energy transfered by the magnetic connection. Let us define a characteristic magnetic field strength by

$$B_0 \equiv \frac{2I}{M}, \tag{22}$$

then we have $\Psi = \overline{\Psi} B_0 M^2$, $T_{HD} = \overline{T} B_0^2 M^3$, $P_{HD} = \overline{P} B_0^2 M^2$, where the dimensionless quantities $\overline{\Psi}$, $\overline{T}$, and $\overline{P}$ depend only on a/M and r'/M. We have calculated $\overline{P}$ for two cases: (1) $r' = r_{ms}$, i.e., the toroidal current flows on the inner boundary of the disk; the radius where the innermost flux surface touches the disk (r_{in}) is shown in Fig. 1; (2) $r' = r_0$ where r_0 is defined by Eq. (12), so that the magnetic field connects the portions of the horizon of the black hole with $0 < \theta < \pi/3$ and $2\pi/3 < \theta < \pi$ to the whole disk from $r = r_{ms}$ to $r = \infty$; the values of r_0/M as a function of a/M are plotted in Fig. 2. The results for $\overline{P} = P/P_0$, where $P_0 \equiv B_0^2 M^2$, are shown in Fig. 5 and Fig. 6, respectively for the two cases.

In Fig. 5, the toroidal electric current flows on the marginally stable orbit (i.e. the inner edge of the disk). The poloidal magnetic field generated by such a cur-
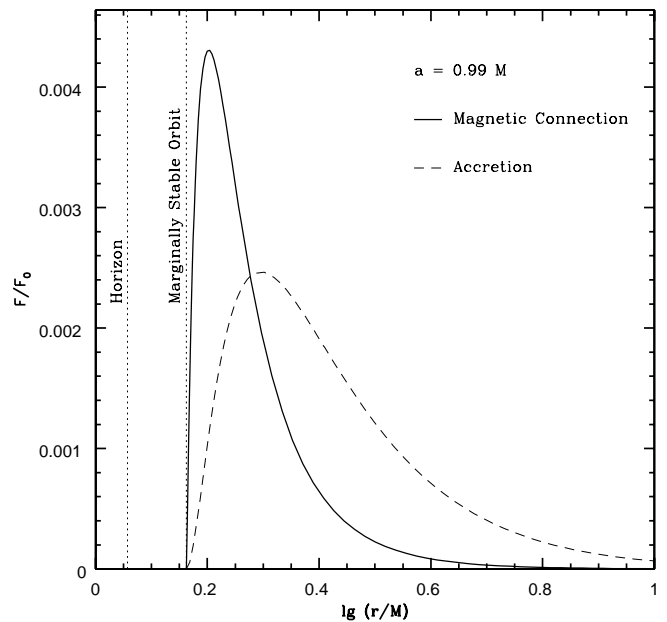
rent connects the horizon of the black hole to the regions of the disk with $r > r_{in}$, where $r_{in} > r_{ms}$ is defined by Eq. (11). Steady state solutions exist only for $a/M \geq 0.01544$. The power produced by the magnetic connection peaks at $a/M = 0.986$ and drops to zero as a/M approaches 1, which is a magnification of the fact that an extreme Kerr black hole expels magnetic fields [20]. If the black hole is spun down from $a/M = 0.998$ to $a/M = 0.01544$, about $0.03Mc^2$ energy is extracted from the black hole. In Fig. 6, the toroidal electric current flows on a circle of radius r_0 defined by Eq. (12). The poloidal magnetic field generated by such a current connects the portions of the horizon of the black hole with $0 < \theta < \pi/3$ and $2\pi/3 < \theta < \pi$ to the whole disk from $r = r_{ms}$ to $r = \infty$. Steady solutions exist only for $a/M \geq 0.3594$. The power produced by the magnetic connection peaks at $a/M = 0.961$ and drops to zero as a/M approaches 1. If the black hole is spun down from $a/M = 0.998$ to $a/M = 0.3594$, about $0.09Mc^2$ energy is extracted from the black hole.

IV. THE RADIATION FLUX OF THE DISK

The energy pumped into the disk by the magnetic connection can be dissipated and radiated away by the disk.

In a steady state without accretion, the radiation flux of the disk, which is the energy radiated per unit time and per unit area by the disk as measured by an observer corotating with the disk, is [4]

$$F = \begin{cases} \frac{1}{r}\left(-\frac{d\Omega_D}{dr}\right)(E^+ - \Omega_D L^+)^{-2}\int_{r_a}^r (E^+ - \Omega_D L^+)\,Hr\,dr\,, & \text{if } r > r_a \\ 0, & \text{if } r < r_a \end{cases}\,, \tag{23}$$

where H is defined by Eq. (18), E^+ and L^+ are the specific energy and the specific angular momentum of a particle in the disk, respectively. For a thin Keplerian disk, we have [5, 6]

$$E^+ = \frac{1 - 2\frac{M}{r} + a\left(\frac{M}{r^3}\right)^{1/2}}{\sqrt{1 - 3\frac{M}{r} + 2a\left(\frac{M}{r^3}\right)^{1/2}}}\,, \tag{24}$$

$$L^+ = (Mr)^{1/2}\,\frac{1 - 2a\left(\frac{M}{r^3}\right)^{1/2} + \frac{a^2}{r^2}}{\sqrt{1 - 3\frac{M}{r} + 2a\left(\frac{M}{r^3}\right)^{1/2}}}\,, \tag{25}$$

and

$$E^+ - \Omega_D L^+ = \frac{\sqrt{1 - 3\frac{M}{r} + 2a\left(\frac{M}{r^3}\right)^{1/2}}}{1 + a\left(\frac{M}{r^3}\right)^{1/2}}\,. \tag{26}$$

Since $r_{ms} \leq r_a$, F is always zero at the inner boundary of the disk. As $r \to \infty$, H declines quickly and the integration in Eq. (23) converges. Thus, as $r \to \infty$, F approaches $\propto r^{-3.5}$.

As in Sec. III, we can calculate the magnetic flux with Eqs. (2 – 7) for a given toroidal current loop at $r = r'$, then determine r_a and the map defined by Eq. (21). Then, with Eqs. (18 – 20), Eq. (23), and Eq. (26), we can calculate the radiation flux of the disk. Using the characteristic magnetic field strength defined by Eq. (22), we have $F = \overline{F}\,F_0$ where $F_0 \equiv B_0^2$, the dimensionless quantity $\overline{F}$ depends on a/M and r'/M only. We have calculated $\overline{F}$ for the following two cases: (1) $r' = r_{ms}$, $a/M = 0.99$; (2) $r' = r_0$, $a/M = 0.99$. The results are shown in Fig. 7 and Fig. 8, respectively. For comparison, we have also shown the radiation flux of a standard thin Keplerian disk (see Appendix A) around the same Kerr black hole in Fig. 7 and Fig. 8.

From Fig. 7 and Fig. 8, we see that the radiation flux produced by the magnetic connection is very different from that produced by the standard accretion: They peak at different radii, and at large radii the radiation flux produced by the magnetic connection declines faster than that produced by the standard accretion. The position of the peak of the radiation flux produced by the magnetic connection depends on the position of the toroidal electric current that generates the poloidal magnetic field. To quantify the variation speed of the radiation flux with radius, we have calculated the emissivity
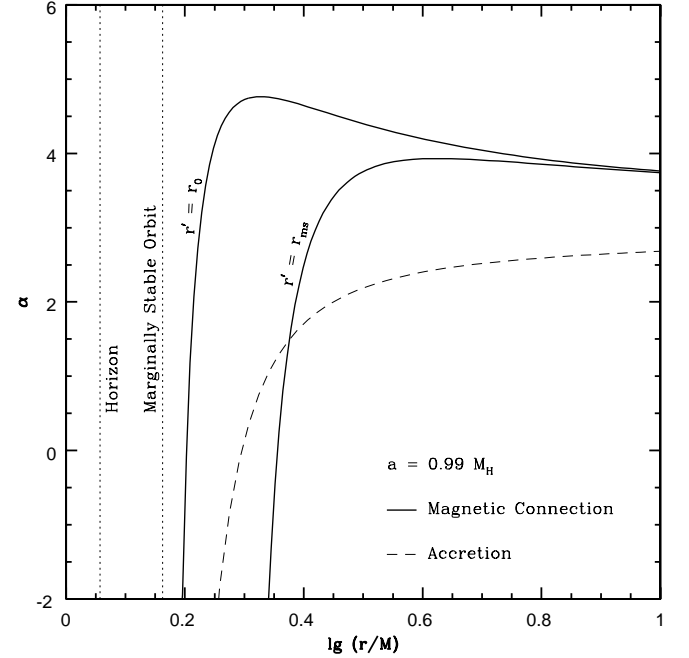

FIG. 9: The emissivity index $\alpha \equiv -d\ln F/d\ln r$ produced by the magnetic connection for the models in Fig. 7 and Fig. 8 (labeled with "$r' = r_{ms}$" and "$r' = r_0$" respectively). For comparison, the emissivity index produced by the standard accretion is shown with the dashed curve.

defined by $\alpha \equiv -d\ln F/d\ln r$, the results are shown in Fig. 9. We see that, the magnetic connection can produce a very large emissivity index compared to the standard accretion, especially when the magnetic field gets more concentrated toward the center of the disk. As $r \to \infty$, the emissivity index of the magnetic connection approaches 3.5, while the emissivity index of accretion approaches 3.

V. CONCLUSIONS

In this paper we have constructed a toy model for the magnetic connection between a black hole and a disk. Assuming a toroidal electric current flows on a circle in the equatorial plane of a Kerr black hole, we have calculated the generated poloidal magnetic field which connects the horizon of the black hole to a thin Keplerian disk around

the black hole, the power produced by the magnetic connection, and the radiation flux of the disk with the assumption of axisymmetry and steady state. We have also compared the results with that of a standard accretion disk.

We have particularly considered two cases: (1) The toroidal electric current flows on the marginally stable orbit where the inner boundary of the disk is. The magnetic field produced by such a toroidal current connects the horizon of the black hole to the disk from $r = r_{in}$ to $r = \infty$, where $r_{in} > r_{ms}$ is defined by Eq. (11). (2) The toroidal electric current flows on a circle of radius r_0 inside the marginally stable orbit, where $r_0 < r_{ms}$ is defined by Eq. (12). The magnetic field produced by such a toroidal current connects the portions of the horizon of the black hole with $0 < \theta < \pi/3$ and $2\pi/3 < \theta < \pi$ to the whole disk from $r = r_{ms}$ to $r = \infty$.

We see that, a magnetic connection between a black hole and a disk is naturally produced by a toroidal current flowing around the black hole in the equatorial plane. The magnetic connection produces a power on the disk without need of accretion, the rotational energy of the black hole provides the energy source. The radiation flux of the disk produced by the magnetic connection is very different from that produced by accretion. Though the position of the peak of the radiation flux depends on the position of the toroidal electric current, at large

radii the radiation flux of magnetic connection always declines faster than that of accretion. Over a large range of radii the emissivity index of magnetic connection is bigger than that of accretion. At large radii the emissivity index of magnetic connection approaches 3.5 from above while the emissivity index of accretion approaches 3 from below.

APPENDIX A: RADIATION FLUX OF A STANDARD ACCRETION DISK

The radiation flux of a standard thin Keplerian disk around a Kerr black hole is [5, 6]

$$F_{acc} = \frac{1}{4\pi r}\dot{M}_D f \,, \qquad (A1)$$

where $\dot{M}_D$ is the mass accretion rate and

$$
\begin{aligned}
f \;=\; & \frac{3}{2M}\frac{1}{x^2\left(x^3 - 3x + 2s\right)}\left[x - x_0 - \frac{3}{2}s\ln\left(\frac{x}{x_0}\right)\right.\\
& -\frac{3\left(x_1 - s\right)^2}{x_1\left(x_1 - x_2\right)\left(x_1 - x_3\right)}\ln\left(\frac{x - x_1}{x_0 - x_1}\right) - \frac{3\left(x_2 - s\right)^2}{x_2\left(x_2 - x_1\right)\left(x_2 - x_3\right)}\ln\left(\frac{x - x_2}{x_0 - x_2}\right)\\
& \left. -\frac{3\left(x_3 - s\right)^2}{x_3\left(x_3 - x_2\right)\left(x_3 - x_1\right)}\ln\left(\frac{x - x_3}{x_0 - x_3}\right)\right] \,,
\end{aligned}
\qquad (A2)
$$

where $s \equiv a/M$, $x \equiv (r/M)^{1/2}$, $x_0 \equiv (r_{ms}/M)^{1/2}$, x_1, x_2, and x_3 are the three roots of $x^3 - 3x + 2s = 0$

$$x_1 \;=\; 2\cos\left(\frac{1}{3}\arccos s - \frac{\pi}{3}\right) \,, \qquad (A3a)$$

$$x_2 \;=\; 2\cos\left(\frac{1}{3}\arccos s + \frac{\pi}{3}\right) \,, \qquad (A3b)$$

$$x_3 \;=\; -2\cos\left(\frac{1}{3}\arccos s\right) \,. \qquad (A3c)$$

It can be checked that $f(r = r_{ms}) = 0$, and $f(r \gg r_{ms}) \approx 3M/2r^2$.

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